

The total irregularity of a graph

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Abstract

In this note a new measure of irregularity of a simple undirected graph G is introduced. It is named the *total irregularity* of a graph and is defined as $\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)|$, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. The graphs with maximal total irregularity are determined. It is also shown that among all trees of same order the star graph has the maximal total irregularity.

1 Introduction

We consider only finite, undirected graphs without loops or multiple edges. For a graph G , we denote by $n = |V(G)|$ and $m = |E(G)|$ its order and size, respectively. For $v \in V(G)$, the degree of v , denoted by $d_G(v)$, is the number of edges incident to v . By $N_G(u)$, we denote the set of vertices that are adjacent to a vertex u , and by $\overline{N}_G(u)$ the set of vertices that are not adjacent to u . A sequence of non-negative integers d_1, \dots, d_n is a *graphic sequence*, or a *degree sequence*, if there exists a graph with the vertex set $\{v_1, \dots, v_n\}$ such that $d(v_i) = d_i$. A *pendant* vertex is a vertex of degree one. A *universal* vertex is the vertex adjacent to all other vertices. A set of vertices is said to be *independent* when the vertices are pairwise non-adjacent. The vertices from an independent set are *independent vertices*.

A graph is *regular* if all its vertices have the same degree, otherwise it is *irregular*. However, it is of interest to measure how irregular it is. Several approaches have been proposed that characterize how irregular a graph is.

Albertson [5] defines the *imbalance* of an edge $e = uv \in E(G)$ as $|d_G(u) - d_G(v)|$ and the *irregularity* of G as

$$\text{irr}(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)|. \quad (1)$$

He presented upper bounds on irregularity for bipartite graphs, triangle-free graphs and arbitrary graphs, as well as a sharp upper bound for trees. Some claims about bipartite graphs given in [5] have been formally proved in [16]. Related to the work of Albertson is the work of Hansen and Mélot [15], who characterized the graphs with n vertices and m edges with maximal irregularity. For more results on imbalance, the irregularity of a graph, and other approaches, that characterize how irregular a graph is, we redirect the reader to [3, 4, 7, 8, 9, 10, 11, 12, 14, 17, 18, 19].

In the sequel we introduce and consider an irregularity measure that is related to the irregularity measure (1). As well as (1), the new measure also captures the irregularity only by a single parameter, namely the degree of a vertex, and for a graph G it is defined as

$$\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)|. \quad (2)$$

Because of the obvious connection with the irregularity of a graph, we called it the *total irregularity* of a graph. Note that the total irregularity of a given graph is completely determined by its degree sequence – graphs with the same degree sequences have the same total irregularity, which is an expected property of an irregularity measure. However, this is not always true with the irregularity of a graph (see Figure 1 for such an example).



Figure 1: Two non-isomorphic graphs G_1 and G_2 with the same degree sequence 1, 1, 1, 1, 2, 3, 3. They have different irregularities ($\text{irr}(G_1) = 10$ and $\text{irr}(G_2) = 8$), but the same total irregularity ($\text{irr}_t(G_1) = \text{irr}_t(G_2) = 22$).

Obviously, both measures are zero if and only if G is regular, and $\text{irr}_t(G)$ is an upper bound of $\text{irr}(G)$. Very recently, these two measurements were compared in [1], where it was shown that for a connected graph G with n vertices, $\text{irr}_t(G) \leq n^2 \text{irr}(G)/4$. Moreover, if G is a tree, then it was shown that $\text{irr}_t(G) \leq (n-2)\text{irr}(G)$. In this note, we focus on graphs with maximal total irregularity.

2 Graphs with maximal total irregularity

Let G_{\max} be a graph with n vertices and with maximal irr_t . Assume that G_{\max} has q universal vertices, where $0 \leq q < n$ (the case $q = n$ is excluded because then $\text{irr}_t(G) = 0$). We denote by U the set of universal vertices of G_{\max} . Let $\bar{U} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{n-q}\}$ the set of non-universal vertices of G_{\max} . We assume that $d(\bar{u}_1) \geq d(\bar{u}_2) \geq \dots \geq d(\bar{u}_{n-q-1}) \geq d(\bar{u}_{n-q})$.

Proposition 2.1. *Let $\bar{u}_i, \bar{u}_j \in \bar{U}$, $i < j$. Then,*

- (a) *there is an edge between \bar{u}_i and \bar{u}_j , if $i + j < n - 2q + 1$;*

(b) there is no edge between \bar{u}_i and \bar{u}_j , if $i + j > n - 2q + 1$;

(c) inserting or deleting an edge $\bar{u}_i\bar{u}_j$ from G_{max} does not change $\text{irr}_t(G_{max})$, if $i + j = n - 2q + 1$.

Proof. (a) Assume that G_{max} does not contain an edge $\bar{u}_i\bar{u}_j$, where $i + j < n + 1 - 2q$. We add such an edge $\bar{u}_i\bar{u}_j$, obtaining a graph G_a . The degrees of both vertices \bar{u}_i and \bar{u}_j increase by one. The change of the total irregularity between \bar{u}_i and the universal vertices is $\sum_{x \in U} |d_{G_a}(x) - d_{G_a}(\bar{u}_i)| - \sum_{x \in U} |d_{G_{max}}(x) - d_{G_{max}}(\bar{u}_i)| = -q$, and the change of the total irregularity between \bar{u}_j and the universal vertices is $\sum_{x \in U} |d_{G_a}(x) - d_{G_a}(\bar{u}_j)| - \sum_{x \in U} |d_{G_{max}}(x) - d_{G_{max}}(\bar{u}_j)| = -q$. For the change of the total irregularity between \bar{u}_i and the vertices in \bar{U} , it holds that $\sum_{\bar{u}_k \in \bar{U}, k < i} |d_{G_a}(\bar{u}_k) - d_{G_a}(\bar{u}_i)| - \sum_{\bar{u}_k \in \bar{U}, k < i} |d_{G_{max}}(\bar{u}_k) - d_{G_{max}}(\bar{u}_i)| \geq -i + 1$, and $\sum_{\bar{u}_k \in \bar{U}, k > i} |d_{G_a}(\bar{u}_k) - d_{G_a}(\bar{u}_i)| - \sum_{\bar{u}_k \in \bar{U}, k > i} |d_{G_{max}}(\bar{u}_k) - d_{G_{max}}(\bar{u}_i)| \geq n - q - i - 1$. Similarly, for the change of the total irregularity between \bar{u}_j and the vertices in \bar{U} , it holds that $\sum_{\bar{u}_k \in \bar{U}, k < j} |d_{G_a}(\bar{u}_k) - d_{G_a}(\bar{u}_j)| - \sum_{\bar{u}_k \in \bar{U}, k < j} |d_{G_{max}}(\bar{u}_k) - d_{G_{max}}(\bar{u}_j)| \geq -j + 2$, and $\sum_{\bar{u}_k \in \bar{U}, k > j} |d_{G_a}(\bar{u}_k) - d_{G_a}(\bar{u}_j)| - \sum_{\bar{u}_k \in \bar{U}, k > j} |d_{G_{max}}(\bar{u}_k) - d_{G_{max}}(\bar{u}_j)| \geq n - q - j$. Thus,

$$\begin{aligned} \text{irr}_t(G_a) &\geq \text{irr}_t(G_{max}) - q - (i - 1) + (n - q - i - 1) - q - (j - 2) + (n - q - j) \\ &= \text{irr}_t(G_{max}) + 2(n - 2q + 1 - i - j) \\ &> \text{irr}_t(G_{max}), \end{aligned}$$

which contradicts the assumption that G_{max} is a graph with maximal irr_t .

(b) Assume that G_{max} contains an edge $\bar{u}_i\bar{u}_j$ such that $i + j > n - 2q + 1$. We delete such an edge $\bar{u}_i\bar{u}_j$, obtaining a graph G_b . Similarly as in (a), we have

$$\begin{aligned} \text{irr}_t(G_b) &\geq \text{irr}_t(G_{max}) + q + (i - 1) - (n - q - i - 1) + q + (j - 2) - (n - q - j) \\ &= \text{irr}_t(G_{max}) + 2(-n + 2q - 1 + i + j) \\ &> \text{irr}_t(G_{max}), \end{aligned}$$

which is a contradiction to the fact that G_{max} is a graph with maximal irr_t .

(c) Assume that G_{max} does not contain an edge $\bar{u}_i\bar{u}_j$ such that $i + j = n - 2q + 1$. We add an edge $\bar{u}_i\bar{u}_j$, where $i + j = n - 2q + 1$, to G_{max} , obtaining a graph G_c . From (a) and (b), it follows that $d(\bar{u}_k)$ is strictly bigger than $d(\bar{u}_i)$, for all $k < i$. Thus, we have

$$\begin{aligned} \text{irr}_t(G_c) &= \text{irr}_t(G_{max}) - q - (i - 1) + (n - q - i - 1) - q - (j - 2) + (n - q - j) \\ &= \text{irr}_t(G_{max}) + 2(n - 2q + 1 - i - j) \\ &= \text{irr}_t(G_{max}). \end{aligned}$$

□

In the sequel, to simplify the notation we denote $N_{G_{max}}(\bar{u}_1) \cup \{\bar{u}_1\}$ by N , and we use \bar{N} instead of $\bar{N}_{G_{max}}(\bar{u}_1)$. By Proposition 2.1, we have that \bar{u}_1 is adjacent to all vertices \bar{u}_i , $i < n - 2q$, it is not adjacent to all vertices \bar{u}_i , $i > n - 2q$, and it might be adjacent to \bar{u}_{n-2q} . Therefore, we have the following corollary.

Corollary 2.1. $|\bar{N}| = q$ or $|\bar{N}| = q + 1$.

Now, we determine the maximal value of irr_t of general graphs.

Theorem 2.1. *For any simple, undirected graph G , $\text{irr}_t(G) \leq \frac{1}{12}(2n^3 - 3n^2 - 2n + 3)$.*

Proof. By Proposition 2.1(c), adding or deleting edges $\bar{u}_i\bar{u}_j$, where $i + j = n - 2q + 1$, does not change $\text{irr}_t(G_{\max})$. Thus, further we consider that G_{\max} does not contain these edges. Then, by Corollary 2.1, it follows that $|\bar{N}| = q + 1$.

The degrees of the vertices in N are as follows: $d(\bar{u}_i) = n - q - 1 - i$, for $i = 1, \dots, \lceil (n - 2q - 1)/2 \rceil$, and $d(\bar{u}_i) = n - q - i$, for $i = \lceil (n - 2q - 1)/2 \rceil + 1, \dots, n - 2q - 1$. All vertices in \bar{N} have degree q . The vertices in U are universal and they have degree $n - 1$.

The contribution between the vertices from U and N to $\text{irr}_t(G_{\max})$ is

$$\begin{aligned} & \sum_{u_i \in U} \sum_{\bar{u}_j \in N} |d(u_i) - d(\bar{u}_j)| \\ &= q \left(\sum_{i=1}^{\lceil \frac{n-2q-1}{2} \rceil} (n-1 - (n-q-1-i)) + \sum_{i=\lceil \frac{n-2q-1}{2} \rceil+1}^{n-2q-1} (n-1 - (n-q-i)) \right) \\ &= \frac{1}{2}q \left((n-2)(n-2q-1) + 2 \left\lceil \frac{n-2q-1}{2} \right\rceil \right). \end{aligned} \quad (3)$$

The contribution between the vertices from U and \bar{N} to $\text{irr}_t(G_{\max})$ is

$$\sum_{u_i \in U} \sum_{\bar{u}_j \in \bar{N}} |d(u_i) - d(\bar{u}_j)| = q(q+1)(n-1-q). \quad (4)$$

The contribution between the vertices from N and \bar{N} is

$$\begin{aligned} & \sum_{\bar{u}_i \in N} \sum_{\bar{u}_j \in \bar{N}} |d(\bar{u}_i) - d(\bar{u}_j)| \\ &= (q+1) \left(\sum_{i=1}^{\lceil \frac{n-2q-1}{2} \rceil} (n-q-1-i-q) + \sum_{i=\lceil \frac{n-2q-1}{2} \rceil+1}^{n-2q-1} (n-q-i-q) \right) \\ &= \frac{1}{2}(q+1) \left((n-2q)(n-2q-1) - 2 \left\lceil \frac{n-2q-1}{2} \right\rceil \right). \end{aligned} \quad (5)$$

Finally, the contribution between the vertices from N to $\text{irr}_t(G_{\max})$ is

$$\begin{aligned} & \sum_{\bar{u}_i \in N} \sum_{\bar{u}_j \in N} |d(\bar{u}_i) - d(\bar{u}_j)| \\ &= \sum_{i=1}^{\lceil \frac{n-2q-1}{2} \rceil-1} \sum_{j=i+1}^{\lceil \frac{n-2q-1}{2} \rceil} (n-q-i-1 - (n-q-j-1)) \\ &+ \sum_{i=1}^{\lceil \frac{n-2q-1}{2} \rceil} \sum_{j=\lceil \frac{n-2q-1}{2} \rceil+1}^{n-2q-1} (n-q-i-1 - (n-q-j)) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=\lceil \frac{n-2q-1}{2} \rceil + 1}^{n-2q-2} \sum_{j=i+1}^{n-2q-1} (n - q - i - (n - q - j)) \\
& = \sum_{i=1}^{\lceil \frac{n-2q-1}{2} \rceil - 1} \sum_{j=i+1}^{\lceil \frac{n-2q-1}{2} \rceil} (j - i) + \sum_{i=1}^{\lceil \frac{n-2q-1}{2} \rceil} \sum_{j=\lceil \frac{n-2q-1}{2} \rceil + 1}^{n-2q-1} (j - i - 1) \\
& + \sum_{i=\lceil \frac{n-2q-1}{2} \rceil + 1}^{n-2q-2} \sum_{j=i+1}^{n-2q-1} (j - i) \\
& = \frac{1}{6}(n - 2q)(n - 2q - 1)(n - 2q - 2) - \left((n - 2q - 1) - \left\lceil \frac{n - 2q - 1}{2} \right\rceil \right) \left\lceil \frac{n - 2q - 1}{2} \right\rceil. \quad (6)
\end{aligned}$$

After simplifying the sum of (3), (4), (5), and (6), we have

$$\text{irr}_t(G_{\max}) = \begin{cases} \frac{1}{12}(2n^3 - 3n^2 - 2n - 4q^3 + 4q) & n \text{ even,} \\ \frac{1}{12}(2n^3 - 3n^2 - 2n - 4q^3 + 4q + 3) & n \text{ odd.} \end{cases} \quad (7)$$

The maxima of the right side expressions in (7) are obtained for $q = 1$. Thus, finally we have

$$\text{irr}_t(G_{\max}) = \begin{cases} \frac{1}{12}(2n^3 - 3n^2 - 2n) & n \text{ even,} \\ \frac{1}{12}(2n^3 - 3n^2 - 2n + 3) & n \text{ odd.} \end{cases}$$

□

In Figure 2, graphs with maximal total irregularity for $n = 4, 5, 6, 7, 8$ are depicted.

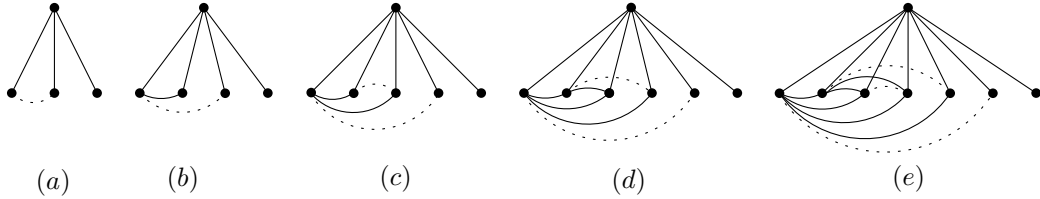


Figure 2: Graphs with maximal total irregularity with (a) 4, (b) 5, (c) 6, (d) 7, (e) 8 vertices. The dotted edges are optional.

There are $\lfloor \frac{n}{2} \rfloor - 1$ optional edges in G_{\max} (edges $\bar{u}_i \bar{u}_j$ that satisfy $i + j = n - 1$ and do not change $\text{irr}_t(G_{\max})$). Thus, the number of graphs of order n with the maximal total irregularity is $2^{\lfloor \frac{n}{2} \rfloor - 1}$.

Proposition 2.2. *Let G be a tree with n vertices. Then, $\text{irr}_t(G) \leq (n - 1)(n - 2)$. Moreover, equality holds if and only if G is a star graph.*

Proof. Let G be a tree that is not a star, with u as a vertex with maximal degree. Consider a pendant vertex v that is not adjacent to u , and is adjacent to a vertex w . We remove the

edge vw and add the edge uv , obtaining a graph G' . After this replacement, only the degrees of u and w change, namely, $d_{G'}(u) = d_G(u) + 1$ and $d_{G'}(w) = d_G(w) - 1$. Thus, we have

$$\begin{aligned} |d_{G'}(u) - d_{G'}(w)| - |d_G(u) - d_G(w)| &= 2, \\ \sum_{x \in V(G) \setminus \{u\}} |d_{G'}(w) - d_{G'}(x)| - \sum_{x \in V(G) \setminus \{u\}} |d_G(w) - d_G(x)| &\geq -n + 2, \quad \text{and} \\ \sum_{x \in V(G) \setminus \{w\}} |d_{G'}(u) - d_{G'}(x)| - \sum_{x \in V(G) \setminus \{w\}} |d_G(u) - d_G(x)| &= n - 1. \end{aligned}$$

From the above relations, we obtain $\text{irr}(G') - \text{irr}(G) = 2 - n + 2 + n - 1 = 3$, and therefore $\text{irr}(G') > \text{irr}(G)$. If G' is not the star, then we repeat the above replacement until the resulting graph is the star. The irregularity of the star graph of order n is $(n-1)(n-2)$. \square

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